Uniqueness of the Solution of Electromagnetic Boundary-Value Problems in the Presence of Lossy and Piecewise Homogeneous Lossless Dielectrics

Salvatore Caorsi and Mirco Raffetto

***Abstract*— In this paper, the uniqueness of the solution of electromagnetic boundary-value problems is investigated and, in some cases, proven. The boundary-value problems considered are always defined in limited regions containing linear dielectric materials that are neither lossy nor lossless everywhere. The boundary conditions are given by specifying over the closed boundaries either the tangential components of the electric field or the tangential components of the magnetic field (or possibly by specifying the tangential components of the electric field over part of the boundaries and the tangential components of the magnetic field over the rest of the boundaries). In particular, the solution is proven to be unique in the case of linear dielectric materials which are piecewise homogeneous and lossless, except for some linear and lossy subregions that may be inhomogeneous. As a byproduct of this analysis, one can conclude that a cavity resonator, loaded with linear and lossy dielectrics together with linear piecewise homogeneous and lossless dielectric materials, does not admit undumped resonances.**

***Index Terms*—Electromagnetic theory.**

I. Introduction

ELECTROMAGNETIC boundary-value problems are characterized by unique time-harmonic solutions when the tangential components of the electric field are specified over the closed boundary of a limited region if the dielectric material contained in that region is linear and lossy everywhere [1]. Boundary conditions can also be specified in terms of the tangential components of the magnetic field over the boundary or even in terms of the tangential components of the electric field over part of the boundary and in terms of the tangential components of the magnetic field over the rest of the boundary. However, the condition on the dissipative behavior of the dielectric material cannot be weakened if we want to retain the validity of the proof given in [1], as this proof of the uniqueness of the solution is based on the energy conservation law, i.e., Poynting's theorem.

On the contrary, if the limited region is characterized by a dielectric material which is linear and lossless everywhere, it is well known that the cavity problem, defined by specifying homogeneous tangential components of the electric field over the boundary (or homogeneous tangential components of the magnetic field over the boundary, or homogeneous tangential components of the electric field over part of the boundary, and homogeneous tangential components of the magnetic field over the rest of the boundary), admits nontrivial modal solutions. For this reason, the electromagnetic boundary-value problem with inhomogeneous boundary conditions can admit an infinite number of solutions, i.e., in general, the solution is not unique.

However, between these two classes of electromagnetic boundary-value problems, there is a "wide" class of problems characterized by linear dielectric materials neither lossy nor lossless everywhere. This is the case, for example, of dielectrically loaded cavity resonators or waveguides, when a lossy linear dielectric fills only part of the investigation domain, the rest being empty and, consequently, linear and lossless. Moreover, this kind of problems can be important even though the modeled physical phenomenon is quite different from a boundary-value problem. For example, scattering simulators based on the so-called hybrid techniques [2], [3] often require the solutions of electromagnetic boundary-value problems, which are defined on linear, but only partly lossy, dielectrics. The uniqueness of the solution of the involved boundary-value problems is of particular importance in order to be sure that the method is able to solve the scattering problem correctly [4]–[6].

A first attempt to fill the gap between problems characterized by linear and everywhere lossy dielectrics and problems characterized by linear and everywhere lossless dielectric materials was made in [7]. In [7], the uniqueness of the solution was proven by using the hypothesis that the lossless part of the dielectric was homogeneous. Now, by using a much more detailed mathematical analysis, we try to generalize that result by allowing the lossless part of the dielectric to present jump discontinuities. In particular, we will prove that the tangential components of the electric field over the boundary (or the tangential components of the magnetic field over the boundary, or the former components over part of the boundary, and the latter components over the rest of the boundary) uniquely determine the solution of the corresponding boundary-value problem when the medium within the boundary is linear, lossless, and piecewise homogeneous, except for some linear and lossy subregions that may be inhomogeneous.

Manuscript received June 18, 1996; revised June 6, 1997.

S. Caorsi is with the Department of Electronics, University of Pavia, I-27100 Pavia, Italy.

M. Raffetto is with the Department of Biophysical and Electronic Engineering, University of Genoa, I-16145 Genoa, Italy.

Publisher Item Identifier S 0018-9480(98)07253-6.

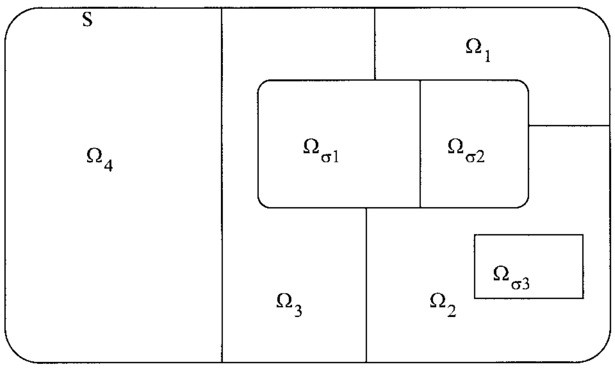

Fig. 1. The boundary-value problem considered involves linear, homogeneous, and lossless materials in $\Omega_i, i=1,\cdots,m$, and linear, but lossy and possibly inhomogeneous dielectrics in $\Omega_{\sigma i}, i=1,\cdots,n$.

A byproduct of this analysis is that a cavity resonator, loaded with linear and lossy dielectrics together with linear piecewise homogeneous and lossless dielectric materials, does not admit undumped resonances.

II. The Electromagnetic Boundary-Value Problem Considered

Fig. 1 shows the typical boundary-value problem considered in this paper. Let Ω be the region of interest. We suppose that it is an open, connected, and bounded set in R^3. Moreover, let $\Omega_i \subset \Omega$, $i=1,\cdots,m$ be open regions characterized by linear, lossless, and homogeneous dielectric materials, and $\Omega_{\sigma i} \subset \Omega$, $i=1,\cdots,n$ be the open regions where the dielectric materials are linear and lossy. This means that the dielectric permittivity and magnetic permeability can be expressed as follows:

$$\varepsilon(\mathbf{r})=\begin{cases}\varepsilon_i, & \mathbf{r}\in\Omega_i,\ i=1,\cdots,m\\ \varepsilon_{\sigma iR}(\mathbf{r})-j\varepsilon_{\sigma iI}(\mathbf{r}), & \mathbf{r}\in\Omega_{\sigma i},\ i=1,\cdots,n\end{cases} \quad (1)$$

$$\mu(\mathbf{r})=\begin{cases}\mu_i, & \mathbf{r}\in\Omega_i,\ i=1,\cdots,m\\ \mu_{\sigma iR}(\mathbf{r})-j\mu_{\sigma iI}(\mathbf{r}), & \mathbf{r}\in\Omega_{\sigma i},\ i=1,\cdots,n\end{cases} \quad (2)$$

where $\varepsilon_i>0$, $i=1,\cdots,m$, $\mu_i>0$, $i=1,\cdots,m$, $\varepsilon_{\sigma iR}(\mathbf{r})$, $\mu_{\sigma iR}(\mathbf{r})$, and $\varepsilon_{\sigma iI}(\mathbf{r})$ or $\mu_{\sigma iI}(\mathbf{r})$ are real scalar fields strictly positive in $\Omega_{\sigma i}$, $i=1,\cdots,n$. Finally, let us suppose that the boundaries S of Ω, S_i of Ω_i, $i=1,\cdots,m$, and $S_{\sigma i}$ of $\Omega_{\sigma i}$, $i=1,\cdots,n$, are piecewise regular surfaces.

The boundary-value problem is mathematically formulated as follows:

given $\omega>0$ and $\mathbf{G}$ find $\mathbf{E}$ and $\mathbf{H}$, defined in every open region Ω_i, $i=1,\cdots,m$, and $\Omega_{\sigma i}$, $i=1,\cdots,n$ (and regular enough to give a meaning to the following equations), satisfying

$$\begin{cases}\nabla\times\mathbf{E}(\mathbf{r})=-j\omega\mu(\mathbf{r})\mathbf{H}(\mathbf{r}), & \text{in } \Omega_i,\ i=1,\cdots,m \\ & \text{or } \Omega_{\sigma_i},\ i=1,\cdots,n\\ \nabla\times\mathbf{H}(\mathbf{r})=j\omega\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}), & \text{in } \Omega_i,\ i=1,\cdots,m\\ & \text{or } \Omega_{\sigma i},\ i=1,\cdots,n\\ \mathbf{n}\times\mathbf{E}=\mathbf{G}, & \text{on } S\end{cases} \quad (3)$$

such that $\mathbf{E}$ and $\mathbf{H}$ are tangentially continuous and $\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$ and $\mu(\mathbf{r})\mathbf{H}(\mathbf{r})$ are normally continuous across internal interfaces.

In the previous problem, ω denotes the angular frequency and $\mathbf{n}$ is the unit vector normal to S (where it has a meaning) and pointing outward from the region Ω.

It is well known that the electromagnetic boundary-value problem (3) admits a unique solution if and only if the corresponding homogeneous problem does not admit nontrivial solutions.

For this reason, in the following, our main objective will be to prove that the associated cavity problem does not admit resonant modes at $\omega>0$. The mathematical formulation of the associated eigenvalue problem is

find $\omega>0$, and nontrivial $\mathbf{E}$ and $\mathbf{H}$, defined in every open region Ω_i, $i=1,\cdots,m$, and $\Omega_{\sigma i}$, $i=1,\cdots,n$ (and regular enough to give a meaning to the following equations), satisfying

$$\begin{cases}\nabla\times\mathbf{E}(\mathbf{r})=-j\omega\mu(\mathbf{r})\mathbf{H}(\mathbf{r}), & \text{in } \Omega_i,\ i=1,\cdots,m \\ & \text{or } \Omega_{\sigma_i},\ i=1,\cdots,n\\ \nabla\times\mathbf{H}(\mathbf{r})=j\omega\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}), & \text{in } \Omega_i,\ i=1,\cdots,m\\ & \text{or } \Omega_{\sigma i},\ i=1,\cdots,n\\ \mathbf{n}\times\mathbf{E}=0, & \text{on } S\end{cases} \quad (4)$$

such that $\mathbf{E}$ and $\mathbf{H}$ are tangentially continuous and $\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$ and $\mu(\mathbf{r})\mathbf{H}(\mathbf{r})$ are normally continuous across internal interfaces.

It is important to note that the boundary conditions in (3) and (4) could also be given in terms of tangential components of the magnetic field ($\mathbf{n}\times\mathbf{H}$ specified on S) or in terms of tangential components of the electric field on part of the boundary and tangential components of the magnetic field over the rest of the boundary ($\mathbf{n}\times\mathbf{E}$ specified on a part S and $\mathbf{n}\times\mathbf{H}$ specified over the rest of S).

III. Some Useful Properties of Electromagnetic Fields

In this section, we collect some important properties of electromagnetic fields, which will be used later to prove our main result on the uniqueness of the solution of the boundary-value problem considered. In particular, by using standard arguments [1] and [8] (i.e., the energy conservation law), Lemma 1 proves that the solutions of the cavity problem (4) are trivial in $\Omega_{\sigma i}$, $i=1,\cdots,n$ (i.e., $\mathbf{E}=\mathbf{H}=0$ in $\Omega_{\sigma i}$, $i=1,\cdots,n$; see also [7] for an analogous, albeit less general, conclusion), Lemma 2 addresses the question of the analyticity of the electromagnetic field in $\Omega_i\subset\Omega$, $i=1,\cdots,m$, and Lemma 3 addresses the problem of the differentiability of the electromagnetic field on the boundary of an open region when this boundary is shared with a region characterized by $\mathbf{E}=\mathbf{H}=0$. The proofs of Lemmas 1 and 3 will be reported in the Appendix. Lemma 2 is proven in [9, p. 134] and [10, pp. 585, 641].

Lemma 1: Any solution $\mathbf{E}$, $\mathbf{H}$ of the cavity problem (4), such that

$$\mathbf{E}\in[C^1(\Theta)]^3 \quad \mathbf{H}\in[C^1(\Theta)]^3, \quad \Theta=\Omega_i,\ i=1,\cdots,m \text{ or } \Theta=\Omega_{\sigma i},;,i=1,\cdots,n \quad (5)$$

satisfies $\mathbf{E}=\mathbf{H}=0$ in $\Omega_{\sigma i}$, $i=1,\cdots,n$.

Lemma 2: In an open region Θ, characterized by constant and possibly complex dielectric parameters, any twice continuously differentiable time-harmonic electromagnetic field, i.e.,

$$\mathbf{E} \in [C^2(\Theta)]^3 \quad \mathbf{H} \in [C^2(\Theta)]^3 \tag{6}$$

is analytic in Θ.

Before continuing, let us introduce the following notation: a complex-valued function defined in Θ, an open set in R^3, is of class $C^{2,\alpha}(\overline{\Theta})$, $\alpha \in (0, 1]$, if $f \in C^2(\Theta)$ and $\exists C > 0$: $\sup_{x,\, y \in \Theta}(|g(x) - g(y)|)/|x-y|^\alpha < C$, where g is the function f itself or any of its first- or second-order partial derivatives [10, p. 674].

Lemma 3: Let Θ be a source-free open set in R^3 and let the dielectric material in Θ be linear. Moreover, let the complex dielectric permittivity $\varepsilon(\mathbf{r})$ and magnetic permeability $\mu(\mathbf{r})$ be twice continuously differentiable, bounded together with their first- and second-order derivatives, and be such that $\exists k > 0$: $|\varepsilon(\mathbf{r})| > k$, $|\mu(\mathbf{r})| > k$, $\forall\, \mathbf{r} \in \Theta$. Any time-harmonic electromagnetic field satisfying

$$\mathbf{E} \in [C^{2,1}(\overline{\Theta})]^3 \quad \mathbf{H} \in [C^{2,1}(\overline{\Theta})]^3 \tag{7}$$

can be extended by continuity on any open regular part Γ of the boundary of Θ if Γ is also a part of the boundary of an open region Θ_σ where $\mathbf{E} = \mathbf{H} = 0$. The extended field is twice continuously differentiable even on Γ, i.e.,

$$\mathbf{E} \in [C^2(\Theta \cup \Gamma \cup \Theta_\sigma)]^3 \quad \mathbf{H} \in [C^2(\Theta \cup \Gamma \cup \Theta_\sigma)]^3. \tag{8}$$

IV. A Uniqueness Theorem

Now we are ready to prove that the cavity problem (4) does not admit resonant modes.

Theorem 1: Any solution of the cavity problem (4) such that

$$\mathbf{E} \in [C^1(\Omega_{\sigma i})]^3 \quad \mathbf{H} \in [C^1(\Omega_{\sigma i})]^3, \qquad i = 1, \cdots, n \tag{9}$$

$$\mathbf{E} \in [C^{2,1}(\overline{\Omega}_i)]^3 \quad \mathbf{H} \in [C^{2,1}(\overline{\Omega}_i)]^3, \qquad i = 1, \cdots, m \tag{10}$$

is trivial, i.e., $\mathbf{E} = \mathbf{H} = 0$ in Ω.

Proof: First, we observe that (9) and (10) imply that (5) is satisfied. Then, by using Lemma 1, we have

$$\mathbf{E} = \mathbf{H} = 0 \text{ in } \Omega_{\sigma i}, \qquad i = 1, \cdots, n. \tag{11}$$

Moreover, by using Lemma 2, we can conclude that E_x, E_y, E_z, H_x, H_y, and H_z are analytic in Ω_i, $i = 1, \cdots, m$.

Now, let $1 \le j \le m$, $1 \le k \le n$, and Ω_j be an open region having part of the boundary in $\overline{\Omega}_{\sigma k}$. Let S_{jk} be an open and regular part of $\overline{\Omega}_j \cap \overline{\Omega}_{\sigma k}$ (see Fig. 2, where we have considered, for example, $j = 1$ and $k = 1$). As a consequence of (10) and (11), the conclusion of Lemma 3 can be applied, and we can define

$$\mathbf{E}' = \begin{cases} \mathbf{E}, & \text{in } \Omega_j \cup \Omega_{\sigma k} \\ 0, & \text{on } S_{jk} \end{cases}$$

and

$$\mathbf{H}' = \begin{cases} \mathbf{H}, & \text{in } \Omega_j \cup \Omega_{\sigma k} \\ 0, & \text{on } S_{jk} \end{cases} \tag{12}$$

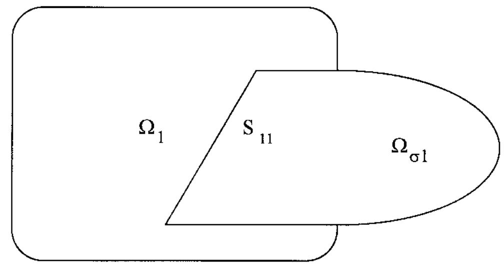

Fig. 2. Two open regions Ω_1 and $\Omega_{\sigma 1}$ are separated by a piecewise regular surface S_{11}. In Ω_1 the dielectric is linear, homogeneous, and lossless. In $\Omega_{\sigma 1}$ it is linear, lossy, and possibly inhomogeneous.

which are $[C^2(B(\mathbf{P}, \delta))]^3$ in an open sphere $B(\mathbf{P}, \delta)$ centered at $\mathbf{P} \in S_{jk}$ of radius δ (δ such that $B(\mathbf{P}, \delta) \subset \Omega_j \cup \Omega_{\sigma k} \cup S_{jk}$). It then follows that $\mathbf{E}'$ and $\mathbf{H}'$ satisfy

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_j \mathbf{H}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon_j \mathbf{E}(\mathbf{r}) \end{cases} \tag{13}$$

in $B(\mathbf{P}, \delta)$, since they satisfy it in $\Omega_j \cap B(\mathbf{P}, \delta)$ by hypothesis, and $\mathbf{E}' = \mathbf{H}' = \nabla \times \mathbf{E}' = \nabla \times \mathbf{H}' = 0$ in $B(\mathbf{P}, \delta) - \Omega_j$ as a consequence of (11) and (12), and of the proven regularity of $\mathbf{E}'$ and $\mathbf{H}'$ on S_{jk}.

However, by virtue of Lemma 2, $\mathbf{E}'$ and $\mathbf{H}'$ are analytic in $B(\mathbf{P}, \delta)$. As $\mathbf{E}' = \mathbf{H}' = 0$ in $\Omega_{\sigma k}$, we obtain $\mathbf{E}' = \mathbf{H}' = 0$ in $B(\mathbf{P}, \delta)$. Finally, by observing that $\Omega_j \cap B(\mathbf{P}, \delta)$ is an open set contained in Ω_j, that $\mathbf{E}' = \mathbf{H}' = 0$ in $B(\mathbf{P}, \delta)$ implies $\mathbf{E} = \mathbf{H} = 0$ in $\Omega_j \cap B(\mathbf{P}, \delta)$, and that $\mathbf{E}$ and $\mathbf{H}$ are analytic in Ω_j, we obtain, by analytic continuation, $\mathbf{E} = \mathbf{H} = 0$ in Ω_j.

This procedure can be applied to every open subregion Ω_i, $i = 1, \cdots, m$, sharing parts of its boundary with a region where $\mathbf{E} = \mathbf{H} = 0$, i.e., we can go across jump discontinuities of the medium, provided that the fields $\mathbf{E}$ and $\mathbf{H}$ satisfy $\mathbf{E} = \mathbf{H} = 0$ in the adjacent region. Therefore, $\mathbf{E} = \mathbf{H} = 0$ in Ω.

Remark 1: Theorem 1 proves that the cavity problem (4) admits only the trivial solution $\mathbf{E} = \mathbf{H} = 0$ within the class of time-harmonic electromagnetic fields with the assumed regularity; we cannot exclude that less regular solutions could exist, but an analysis of this problem would require the use of the tools of functional analysis.

Remark 2: Theorem 1 could also be proven by using Lemma 1 and repeatedly applying a result proven by Müller [9, Th. 34, p. 135]. However, this result does not provide the regular (C^2) continuation property proven in Lemma 3.

An immediate consequence of Theorem 1 is that boundary-value problem (3) has a unique solution.

V. Conclusions

A generalization of the standard uniqueness theorem for time-harmonic electromagnetic boundary-value problems has been presented and proven. The boundary-value problems considered have been defined by specifying the tangential components of the electric field over the closed boundary (or the tangential components of the magnetic field over the boundary, or the former components over part of the boundary, and the latter components over the rest of the boundary) of a limited region containing a linear dielectric material neither lossy nor lossless everywhere. In particular, the uniqueness of the solution has been proven in the case where the dielectric was linear, piecewise, homogeneous, and lossless everywhere, except for some linear and lossy subregions that may be inhomogeneous. From this analysis, one can deduce that a cavity resonator, loaded with linear and lossy dielectrics together with linear, piecewise, homogeneous, and lossless dielectric materials, does not admit undumped resonances.

APPENDIX

In this appendix, some mathematical details and the proofs of Lemmas 1 and 3 are provided.

In the boundary-value problem and the corresponding eigenvalue problem considered in this paper, we look for solutions defined in open regions Ω_i, $i = 1, \cdots, m$, and $\Omega_{\sigma i}$, $i = 1, \cdots, n$. For this reason, the boundary conditions must be considered as conditions on the limits of the fields as they approach the boundary from the interior of the domain of definition [10, vol. I, p. 579]. Moreover, we required some regularity of the fields in those regions in order to give a meaning to the equations or in order to prove some properties of the fields themselves. Then, the boundary condition $\mathbf{n}\times\mathbf{E} = \mathbf{G}$ on S can have a meaning if $\mathbf{G}$ is not too irregular [10, p. 578].

Now we can turn our attention to the proofs of the lemmas.

Proof of Lemma 1: The continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$ across the possible internal interfaces implies that $\mathbf{E}\times\mathbf{H}^*$ (* indicates complex conjugate) has a continuous normal component. Moreover, $\mathbf{E}\times\mathbf{H}^*$ is continuously differentiable in every open subregion, and the boundary of every subregion is piecewise regular. Consequently, the divergence theorem can be applied to the whole domain Ω as

$$\oint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{n}\,dS = \int_\Omega \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)\,dV. \tag{A1}$$

By using the vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{A2}$$

we obtain

$$\oint_S \mathbf{E}\times\mathbf{H}^*\cdot n\,dS = \int_\Omega \mathbf{H}^*\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}^*)\,dV \tag{A3}$$

and, by using Maxwell's equations, we have

$$\begin{aligned}\oint_S &\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{n}\,dS \\ &= \int_\Omega \mathbf{H}^*\cdot(-j\omega\mu(\mathbf{r})\mathbf{H}) - \mathbf{E}\cdot(-j\omega\varepsilon^*(\mathbf{r})\mathbf{E}^*)\,dV \\ &= \int_\Omega -j\omega\mu(\mathbf{r})|\mathbf{H}|^2 + j\omega\varepsilon^*(\mathbf{r})|\mathbf{E}|^2\,dV. \end{aligned}\tag{A4}$$

Since $\mathbf{n}\times\mathbf{E} = 0$ over S (or $\mathbf{n}\times\mathbf{H} = 0$ over S, or $\mathbf{n}\times\mathbf{E} = 0$ over part of S and $\mathbf{n}\times\mathbf{H} = 0$ over the rest of S), we have

$$\oint_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{n}\,dS = 0 \tag{A5}$$

and, consequently,

$$0 = \int_\Omega -j\omega\mu(\mathbf{r})|\mathbf{H}|^2 + j\omega\varepsilon^*(\mathbf{r})|\mathbf{E}|^2\,dV. \tag{A6}$$

Substituting for $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ from (1) and (2), we obtain

$$\begin{aligned}0 = \sum_{i=1}^{m}\int_{\Omega_i} &-j\omega\mu_i|\mathbf{H}|^2 + j\omega\varepsilon_i|\mathbf{E}|^2\,dV \\ &+ \sum_{i=1}^{n}\int_{\Omega_{\sigma i}} -j\omega(\mu_{\sigma iR}(\mathbf{r}) - j\mu_{\sigma iI}(\mathbf{r}))|\mathbf{H}|^2 \\ &+ j\omega(\varepsilon_{\sigma iR}(\mathbf{r}) + j\varepsilon_{\sigma iI}(\mathbf{r}))|\mathbf{E}|^2\,dV \end{aligned}\tag{A7}$$

i.e.,

$$\begin{aligned}0 = \sum_{i=1}^{m}\int_{\Omega_i} &-j\omega\mu_i|\mathbf{H}|^2 + j\omega\varepsilon_i|\mathbf{E}|^2\,dV \\ &+ \sum_{i=1}^{n}\int_{\Omega_{\sigma i}} -j\omega\mu_{\sigma iR}(\mathbf{r})|\mathbf{H}|^2 + j\omega\varepsilon_{\sigma iR}(\mathbf{r})|\mathbf{E}|^2\,dV \\ &- \sum_{i=1}^{n}\int_{\Omega_{\sigma i}} \omega\mu_{\sigma iI}(\mathbf{r})|\mathbf{H}|^2 + \omega\varepsilon_{\sigma iI}(\mathbf{r})|\mathbf{E}|^2\,dV. \end{aligned}\tag{A8}$$

This equation implies that both the real and imaginary parts of the right-hand-side term are zero. In particular, the real part must be zero, i.e.,

$$\sum_{i=1}^{n}\int_{\Omega_{\sigma i}} \omega\mu_{\sigma iI}(\mathbf{r})|\mathbf{H}|^2 + \omega\varepsilon_{\sigma iI}(\mathbf{r})|\mathbf{E}|^2\,dV = 0. \tag{A9}$$

Both terms of the integrand of (A9) are greater than or equal to zero $\forall\,\mathbf{r}\in\Omega_{\sigma i}$, $i = 1, \cdots, n$. As a consequence, (A9) is satisfied if and only if

$$\begin{cases}\omega\varepsilon_{\sigma iI}(\mathbf{r})|\mathbf{E}|^2 = 0 & \forall\,\mathbf{r}\in\Omega_{\sigma i},\ i = 1, \cdots, n \\ \omega\mu_{\sigma iI}(\mathbf{r})|\mathbf{H}|^2 = 0 & \forall\,\mathbf{r}\in\Omega_{\sigma i},\ i = 1, \cdots, n.\end{cases}\tag{A10}$$

Thus, in order to satisfy (A9), we must have $\mathbf{E} = 0$ where $\omega\varepsilon_{\sigma iI}(\mathbf{r})$ is strictly positive, and $\mathbf{H} = 0$ where $\omega\mu_{\sigma iI}(\mathbf{r})$ is strictly positive.

However, by using Maxwell's equations

$$\begin{cases}\nabla\times\mathbf{E}(r) = -j\omega\mu(\mathbf{r})\mathbf{H}(\mathbf{r}) \\ \nabla\times\mathbf{H}(\mathbf{r}) = j\omega\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\end{cases}\tag{A11}$$

and

$$\begin{cases}-j\omega\mu(\mathbf{r}) \neq 0 \\ j\omega\varepsilon(\mathbf{r}) \neq 0\end{cases} \quad \forall\,\mathbf{r}\in\left(\bigcup_{i=1}^{m}\Omega_i\right)\cup\left(\bigcup_{i=1}^{n}\Omega_{\sigma i}\right) \tag{A12}$$

we have that $\mathbf{E} = 0$ ($\mathbf{H} = 0$) in any subregion of Ω implies $\mathbf{H} = 0$ ($\mathbf{E} = 0$) in the same subregion. Consequently, $\mathbf{E} = \mathbf{H} = 0$ where $\omega\varepsilon_{\sigma iI}(\mathbf{r})$ or $\omega\mu_{\sigma iI}(\mathbf{r})$ are strictly positive. Then, under the hypothesis on the signs of ω, $\varepsilon_{\sigma iI}(\mathbf{r})$ and $\mu_{\sigma iI}(\mathbf{r})$, we obtain

$$\mathbf{E} = \mathbf{H} = 0 \text{ in } \Omega_{\sigma i}, \qquad i = 1, \cdots, n. \tag{A13}$$

Proof of Lemma 3: First of all, we observe that

$$\lim_{r\in\Theta\to P\in\Gamma}\mathbf{E} = \lim_{r\in\Theta\to P\in\Gamma}\mathbf{H} = 0 \tag{A14}$$

as a consequence of the continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and of the normal components of $\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$ and $\mu(\mathbf{r})\mathbf{H}(\mathbf{r})$ (as $|\varepsilon(\mathbf{r})| > k$ and $|\mu(\mathbf{r})| > k\ \forall\,\mathbf{r}\in\Theta$).

The rest of the proof is divided into three parts. In Part 1, we prove that the tangential derivatives of any component of $\mathbf{E}$ or $\mathbf{H}$ tend to zero as $\mathbf{r}\in\Theta\to\mathbf{P}\in\Gamma$, in Part 2, by

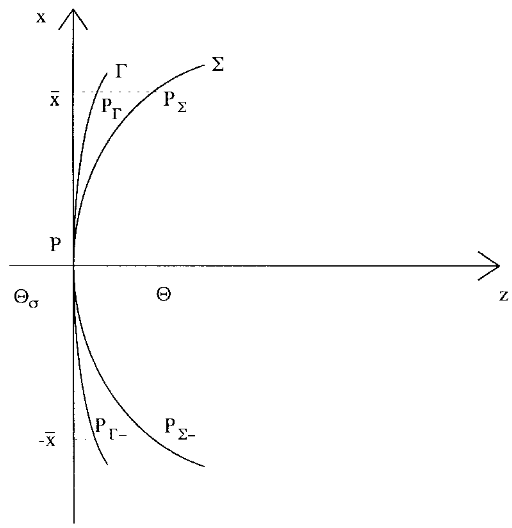

Fig. 3. The chosen Cartesian coordinates are such that the plane $z = 0$ is tangent to Γ at P. In particular, it is shown the plane $y = 0$. Σ is a curve contained, at least locally, in Θ.

using the result of Part 1, we prove that any first- or second-order derivative of any component of $\mathbf{E}$ or $\mathbf{H}$ tend to zero as $\mathbf{r} \in \Theta \to \mathbf{P} \in \Gamma$. Finally, in Part 3, we conclude the proof by using the result of Part 2.

Part 1: Let us consider a point $\mathbf{P}$ of Γ. As Γ is regular, we can define a reference system with origin in $\mathbf{P}$ such that (x, y) is the plane tangent to the surface Γ at $\mathbf{P}$ and z is the axis orthogonal to Γ at $\mathbf{P}$. The z-axis is oriented in such a way that the points $(0, 0, \xi)$, $\xi > 0$ are in Θ. Let us consider the plane $y = 0$, and in this plane the curve Σ defined by $z = |x|^\alpha$, $\alpha \in (1, 2)$. As the boundary Γ is C^2, there exists $\delta > 0$ such that Σ is in Θ for $|x| < \delta$ (see Fig. 3).

Now, let us consider a complex function f, which could be any of the components of $\mathbf{E}$ or $\mathbf{H}$.

Given $0 < \overline{x} < \delta$, the point on Γ corresponding to $x = \overline{x}$, $y = 0$ is given by $\mathbf{P}_\Gamma = (\overline{x}, 0, (\partial^2 z/\partial x^2)(x_0)(\overline{x}^2/2))$, where $x_0 \in (0, \overline{x})$ and $(\partial^2 z/\partial x^2)(x)$ is the second-order derivative of the curve given by intersecting Γ with the plane $y = 0$. Then, the distance between the point on Γ $\mathbf{P}_\Gamma$ and the point on Σ $\mathbf{P}_\Sigma = (\overline{x}, 0, |\overline{x}|^\alpha)$ is smaller than $|(\partial^2 z/\partial x^2)(x_0)(\overline{x}^2/2)| + |\overline{x}|^\alpha$.

Let us define the function g as follows:

$$g = \begin{cases} f, & \text{in } \Theta \\ 0, & \text{on } \Gamma \end{cases} \tag{A15}$$

which is continuous in $\Theta \cup \Gamma$ as a consequence of (A14).

It is now easy to see that if C is such that $\sup_{x,\, y \in \Theta}(|f(x) - f(y)|/|x-y|) < C$ then $|\nabla f(Q)| \le C \;\forall Q \in \Theta$. If this was not the case, by using the continuity of the gradient and Lagrange's theorem, we could find $x, y \in \Theta$ such that $|f(x) - f(y)| > C|x - y|$.

By using again Lagrange's theorem and applying it to g, we can conclude

$$\frac{|g(\mathbf{P}_\Sigma) - g(\mathbf{P}_\Gamma)|}{|\mathbf{P}_\Sigma - \mathbf{P}_\Gamma|} \le C.$$

This implies

$$|g(\mathbf{P}_\Sigma)| \le C|\mathbf{P}_\Sigma - \mathbf{P}_\Gamma| \le C\left(\left|\frac{\partial^2 z}{\partial x^2}(x_0)\,\frac{\overline{x}^2}{2}\right| + |\overline{x}|^\alpha\right). \tag{A16}$$

In an analogous way, by considering the points on Γ and Σ corresponding to $x = -\overline{x}$ and $y = 0$ (see Fig. 3), i.e.,

$$\mathbf{P}_{\Gamma-} = \left(-\overline{x},\, 0,\, \frac{\partial^2 z}{\partial x^2}(x_{0-})\,\frac{\overline{x}^2}{2}\right)$$

and

$$\mathbf{P}_{\Sigma-} = (-\overline{x},\, 0,\, |\overline{x}|^\alpha)$$

where $x_{0-} \in (-\overline{x}, 0)$, we obtain

$$|g(\mathbf{P}_{\Sigma-})| \le C|\mathbf{P}_{\Sigma-} - \mathbf{P}_{\Gamma-}| \le C\left(\left|\frac{\partial^2 z}{\partial x^2}(x_{0-})\,\frac{\overline{x}^2}{2}\right| + |\overline{x}|^\alpha\right). \tag{A17}$$

Then,

$$C\left(\left|\frac{\partial^2 z}{\partial x^2}(x_0)\,\frac{\overline{x}^2}{2}\right| + \left|\frac{\partial^2 z}{\partial x^2}(x_{0-})\,\frac{\overline{x}^2}{2}\right| + 2|\overline{x}|^\alpha\right) \ge |g(\mathbf{P}_\Sigma)| + |g(\mathbf{P}_{\Sigma-})| \ge |g(\mathbf{P}_\Sigma) - g(\mathbf{P}_{\Sigma-})|. \tag{A18}$$

As $\mathbf{P}_\Sigma$ and $\mathbf{P}_{\Sigma-}$ are in Θ we have [see (A15)]

$$|g(\mathbf{P}_\Sigma) - g(\mathbf{P}_{\Sigma-})| = |f(\mathbf{P}_\Sigma) - f(\mathbf{P}_{\Sigma-})| \tag{A19}$$

and, consequently, by also using Lagrange's theorem,

$$C\left(\left|\frac{\partial^2 z}{\partial x^2}(x_0)\,\frac{\overline{x}^2}{2}\right| + \left|\frac{\partial^2 z}{\partial x^2}(x_{0-})\,\frac{\overline{x}^2}{2}\right| + 2|\overline{x}|^\alpha\right) \ge |f(\mathbf{P}_\Sigma) - f(\mathbf{P}_{\Sigma-})| = \left|\frac{\partial f}{\partial x}(x_1, 0, |\overline{x}|^\alpha)\cdot 2|\overline{x}|\right| \tag{A20}$$

where $x_1 \in (-\overline{x}, \overline{x})$, i.e.,

$$\left|\frac{\partial f}{\partial x}(x_1, 0, |\overline{x}|^\alpha)\right| \le \frac{C}{2}\left(\left|\frac{\partial^2 z}{\partial x^2}(x_0)\,\frac{|\overline{x}|}{2}\right| + \left|\frac{\partial^2 z}{\partial x^2}(x_{0-})\,\frac{|\overline{x}|}{2}\right| + 2|\overline{x}|^{\alpha-1}\right)_{|\overline{x}|\to 0} \to 0. \tag{A21}$$

Let us now suppose $\partial f/\partial x$ does not tend to zero as $r \in \Theta \to \mathbf{P} \in \Gamma$, i.e.,

$$\exists \varepsilon_0\, \forall \overline{\delta} \exists \mathbf{P}_{\overline{\delta}} \in \Theta \cap B(0, \overline{\delta})\colon \left|\frac{\partial f}{\partial x}(\mathbf{P}_{\overline{\delta}})\right| \ge \varepsilon_0 \tag{A22}$$

where $B(0, \overline{\delta})$ denotes the open sphere centered at zero of radius $\overline{\delta}$.

Let us consider $|\overline{x}| = 1/n$; then (A23), shown at the bottom of the following page, contradicts the hypothesis $f \in C^{2,1}(\overline{\Theta})$.

Then, $\partial f/\partial x$ tends to zero as $\mathbf{r} \in \Theta \to \mathbf{P} \in \Gamma$. Analogously, we can prove that all tangential derivatives tend to zero as $\mathbf{r} \in \Theta \to \mathbf{P} \in \Gamma$.

Part 2: From Part 1, we know that $\partial E_x/\partial x$, $\partial E_y/\partial x$, $\partial E_z/\partial x$, $\partial E_x/\partial y$, $\partial E_y/\partial y$, $\partial E_z/\partial y$, $\partial H_x/\partial x$, $\partial H_y/\partial x$, $\partial H_z/\partial x$, $\partial H_x/\partial y$, $\partial H_y/\partial y$, $\partial H_z/\partial y$ tend to zero as

$$\mathbf{r} \in \Theta \rightarrow \mathbf{P} \in \Gamma. \tag{A24}$$

In order to prove that the normal derivatives also tend to zero, let us consider Maxwell's equations and, in particular,

$$\begin{aligned} \frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z &= j\omega\varepsilon E_y \\ \frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y &= j\omega\varepsilon E_x. \end{aligned} \tag{A25}$$

E_x and E_y tend to zero as a consequence of (A14). As ε is bounded, the right-hand sides tend to zero. However, from (A24), we know that $(\partial/\partial x)H_z$ and $(\partial/\partial y)H_z$ tend to zero. Consequently, $(\partial/\partial z)H_x$ and $(\partial/\partial z)H_y$ tend to zero.

By applying the divergence to one of Maxwell's equations

$$\nabla \cdot (\nabla \times \mathbf{E}(\mathbf{r})) = -j\omega \nabla \cdot (\mu(\mathbf{r})\mathbf{H}(\mathbf{r})) \tag{A26}$$

we obtain (as $\mathbf{E}$ is twice continuously differentiable we have $\nabla \cdot (\nabla \times \mathbf{E}(\mathbf{r})) = 0$)

$$\begin{aligned} 0 &= \nabla \cdot (\mu \mathbf{H}) \\ &= \mathbf{H} \cdot \nabla\mu + \mu \nabla \cdot \mathbf{H} \\ &= \mathbf{H} \cdot \nabla\mu + \mu \frac{\partial}{\partial x} H_x + \mu \frac{\partial}{\partial y} H_y + \mu \frac{\partial}{\partial z} H_z. \end{aligned} \tag{A27}$$

However, $\mathbf{H} \cdot \nabla\mu$, $\mu(\partial/\partial x)H_x$, and $\mu(\partial/\partial y)H_y$ tend to zero, as μ and its gradient are bounded. Then, as $|\mu(\mathbf{r})| > k \, \forall \mathbf{r} \in \Theta$, $(\partial/\partial z)H_z$ tends to zero.

Analogously, $(\partial/\partial z)E_x$, $(\partial/\partial z)E_y$, and $(\partial/\partial z)E_z$ tend to zero as $\mathbf{r} \in \Theta \rightarrow \mathbf{P} \in \Gamma$.

We now know that all first-order derivatives of any component of $\mathbf{E}$ or $\mathbf{H}$ tend to zero as $\mathbf{r} \in \Theta \rightarrow \mathbf{P} \in \Gamma$.

By using the same procedure as indicated in Part 1, we obtain that $\partial f/\partial x$, $\partial f/\partial y$ tend to zero, where, in this case, f is any of the first-order derivatives of any component of $\mathbf{E}$ or $\mathbf{H}$. Moreover, by using Schwartz's theorem, any of the following derivatives: $\partial^2 g/(\partial z\, \partial x) = \partial^2 g/(\partial x\, \partial z)$, $\partial^2 g/(\partial z\, \partial y) = \partial^2 g/(\partial y\, \partial z)$ tend to zero, where, in this case, g is any component of $\mathbf{E}$ or $\mathbf{H}$. It remains to be proven that $\partial^2 g/\partial z^2$ tends to zero, where g is again any component of $\mathbf{E}$ or $\mathbf{H}$.

By considering, for example, the derivative with respect to z of $(\partial/\partial z)H_x - (\partial/\partial x)H_z = j\omega\varepsilon E_y$, we obtain

$$\frac{\partial^2}{\partial z^2} H_x - \frac{\partial^2}{\partial z\, \partial x} H_z = j\omega \left(E_y \frac{\partial \varepsilon}{\partial z} + \varepsilon \frac{\partial E_y}{\partial z} \right). \tag{A28}$$

The right-hand side tends to zero (also because ε and its derivatives are bounded in Θ), as does the second addend of the left-hand side. Then, $(\partial^2/\partial z^2)H_x$ tends to zero. Analogously, we can prove $(\partial^2/\partial z^2)H_y$.

We have already noted that $\nabla \cdot (\mu \mathbf{H}) = 0$. As both μ and $\mathbf{H}$ are at least twice continuously differentiable in Θ, we can derive it with respect to z

$$\begin{aligned} 0 &= \frac{\partial}{\partial z} (\nabla \cdot (\mu \mathbf{H})) \\ &= \frac{\partial}{\partial z} (\mathbf{H} \cdot \nabla\mu + \mu \nabla \cdot \mathbf{H}) \\ &= \frac{\partial \mathbf{H}}{\partial z} \cdot \nabla\mu + \mathbf{H} \cdot \frac{\partial}{\partial z} (\nabla\mu) + \frac{\partial \mu}{\partial z} \nabla \cdot \mathbf{H} \\ &\quad + \mu \frac{\partial}{\partial z} \frac{\partial}{\partial x} H_x + \mu \frac{\partial}{\partial z} \frac{\partial}{\partial y} H_y + \mu \frac{\partial}{\partial z} \frac{\partial}{\partial z} H_z. \end{aligned} \tag{A29}$$

We have already proven that the first five addends of the right-hand-side term tend to zero (also because μ and its first- and second-order derivatives are bounded in Θ).

Then, as $|\mu(\mathbf{r})| > k \; \forall \mathbf{r} \in \Theta$, $(\partial^2/\partial z^2)H_z$ also tends to zero as $\mathbf{r} \in \Theta \rightarrow \mathbf{P} \in \Gamma$.

Analogously, we can prove that $(\partial^2/\partial z^2)E_x$, $(\partial^2/\partial z^2)E_y$, $(\partial^2/\partial z^2)E_z$ tend to zero as $\mathbf{r} \in \Theta \rightarrow \mathbf{P} \in \Gamma$.

Part 3: Let us consider

$$\mathbf{E}' = \begin{cases} \mathbf{E}, & \text{in } \Theta \cup \Theta_\sigma \\ 0, & \text{on } \Gamma \end{cases} \qquad \mathbf{H}' = \begin{cases} \mathbf{H}, & \text{in } \Theta \cup \Theta_\sigma \\ 0, & \text{on } \Gamma \end{cases}$$

which is continuous in $\Theta \cup \Gamma \cup \Theta_\sigma$. Let f be any component of $\mathbf{E}'$ or $\mathbf{H}'$. By using the fact that the first-order derivatives of f tend to zero as $\mathbf{r} \in \Theta \rightarrow \mathbf{P} \in \Gamma$, we first prove that f is differentiable on Γ and that its differential is zero on Γ, i.e., $\lim_{\mathbf{r}\rightarrow\mathbf{P}} (f(\mathbf{r}) - f(\mathbf{P})/|\mathbf{r} - \mathbf{P}|) = 0$, $\forall \mathbf{P} \in \Gamma$ or, equivalently,

$$\forall \varepsilon > 0, \exists \delta > 0\!: \left| \frac{f(\mathbf{r}) - f(\mathbf{P})}{|\mathbf{r} - \mathbf{P}|} \right| < \varepsilon \; \forall \mathbf{r}\!: 0 < |\mathbf{r} - \mathbf{P}| < \delta, \quad \forall \mathbf{P} \in \Gamma. \tag{A30}$$

If $\mathbf{r} \in \Gamma \cup \Theta_\sigma$, the inequality clearly holds true. Let $B(\mathbf{P}, \delta)$ be the open sphere centered at $\mathbf{P} \in \Gamma$ of radius δ such that $|\nabla f| < \varepsilon \; \forall \mathbf{x} \in \Theta \cap B(\mathbf{P}, \delta)$ and $B(\mathbf{P}, \delta) \subset \Theta \cup \Gamma \cup \Theta_\sigma$. If $\mathbf{r} \in \Theta \cap B(\mathbf{P}, \delta)$, let us consider the straight line $\overline{\mathbf{rP}}$ ($\mathbf{r} \in \overline{\mathbf{rP}}$ and $\mathbf{P} \in \overline{\mathbf{rP}}$). Only two possible cases are to be considered (see Fig. 4):

- $\overline{\mathbf{rP}}$ and Γ have only $\mathbf{P}$ in common. Then, by using Lagrange's theorem,

$$\exists \mathbf{q}\!: f(\mathbf{r}) = f(\mathbf{P}) + \nabla f(\mathbf{q}) \cdot (\mathbf{r} - \mathbf{P}), \quad \mathbf{q} \in \overline{\mathbf{rP}}, \; \mathbf{q} \neq P, \; \mathbf{q} \neq \mathbf{r} \tag{A31}$$

  and the conclusion is obvious since $|\nabla f(\mathbf{q})| < \varepsilon$.
- $\overline{\mathbf{rP}}$ and Γ have some other points in common. Let $\mathbf{P}_1$ be the closest of these points to $\mathbf{r}$ (it exists as $\overline{\mathbf{rP}}$ and

$$\frac{\left| \frac{\partial f}{\partial x} (\mathbf{P}_{\overline{\delta}}) - \frac{\partial f}{\partial x} (x_1, 0, |\overline{x}|^\alpha) \right|}{|\mathbf{P}_{\overline{\delta}} - (x_1, 0, |\overline{x}|^\alpha)|} > \frac{\varepsilon_0 - \frac{C}{2} \left(\left| \frac{\partial^2 z}{\partial x^2} (x_0) \frac{1}{2n} \right| + \left| \frac{\partial^2 z}{\partial x^2} (x_{0-}) \frac{1}{2n} \right| + 2 \left| \frac{1}{n} \right|^{\alpha - 1} \right)}{\left(\overline{\delta} + \frac{1}{n} \sqrt{1 + \left(\frac{1}{n} \right)^{2\alpha - 2}} \right)} \xrightarrow[\overline{\delta} \rightarrow 0,\, n \rightarrow +\infty]{} +\infty \tag{A23}$$

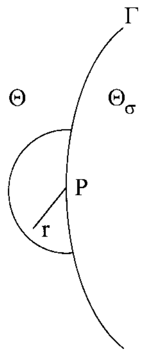

Fig. 4. A neighborhood of a point P belonging to the common boundary of two adjacent regions.

the part of Γ in a sufficiently small closed ball are closed sets). Then, by using Lagrange's theorem

$$\exists \mathbf{q}\text{: } f(\mathbf{r}) = f(\mathbf{P}_1) + \nabla f(\mathbf{q}) \cdot (\mathbf{r} - \mathbf{P}_1), \quad \mathbf{q} \in \overline{\mathbf{r}\mathbf{P}_1},\ \mathbf{q} \neq \mathbf{P}_1,\ \mathbf{q} \neq \mathbf{r}. \tag{A32}$$

Then, observing that $f(\mathbf{P}) = f(\mathbf{P}_1) = 0$ and that $|\mathbf{r} - \mathbf{P}_1| < |\mathbf{r} - \mathbf{P}|$, the conclusion is once again obvious (as $|\nabla f(\mathbf{q})| < \varepsilon$).

Now, by considering as f any derivative of any component of $\mathbf{E}$ or $\mathbf{H}$, we can again use the same procedure to prove that it is differentiable on Γ and that its differential is zero on Γ. Then, $\mathbf{E} \in [C^2(\Theta \cup \Gamma \cup \Theta_\sigma)]^3$, $\mathbf{H} \in [C^2(\Theta \cup \Gamma \cup \Theta_\sigma)]^3$.

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Salvatore Caorsi received the *laurea* degree in electronic engineering from the University of Genoa, Genoa, Italy, in 1973.

He remained with the University of Genoa as a Researcher. In 1976, he was with the Department of Biophysical and Electronic Engineering, as an Associate Professor of antennas and propagation. In 1985, he became a Professor of fundamentals of remote sensing. Since 1994, he has been with the Department of Electronics, University of Pavia, Pavia, Italy, as a full Professor of electromagnetic compatibility. His primary activities are focused on applications of electromagnetic fields to telecommunications, artificial vision and remote sensing, biology, and medicine. In particular, he works on research projects concerning microwave hyperthermia and radiometry in oncological therapy, numerical methods for solving electromagnetic problems, wave propagation in the presence of nonlinear media, and inverse scattering and microwave imaging. He is the Past-President and Funding Member of the Inter-University Research Center for Interactions between Electromagnetic Fields and Biological Systems (ICEmB).

Dr. Caorsi is a member of the Asssociazione Elettrotecnica Italiana (AEI), the European Bioelectromagnetism Association (EBEA), and the European Society for Hyperthermic Oncology (ESHO).

Mirco Raffetto received the *laurea* degree (*summa cum laude*) in electronic engineering, and the Ph.D. degree in models, methods, and tools for electronic and electromagnetic systems from the University of Genoa, Genoa, Italy, in 1990 and 1997, respectively.

He is currently a Research Assistant in the Department of Biophysical and Electronic Engineering, University of Genoa. His main interests are in the fields of electromagnetic scattering, theoretical electromagnetics, and numerical methods in electromagnetics.